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A New Approach to the Theory of Elementary Domains

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It is shown that Yukawa's theory of elementary domains can be formulated in a general framework. A quantized measure space structure of a space-time manifold is introduced so as to represent faithfully the elementary-domain structure. For a realization of elementary domains, an operator valued measure is defined such that it represents the spatio-temporal distribution of elementary domains. Effects of such a quantized topology are illustrated in the expressions of S matrices.

1. INTRODUCTION

We are so used to the macroscopic space-time structure that even when we have to deal with the subnuclear region of space and time we are apt to adhere to classical concepts of space-time such as Minkowski space. However, as is well known, physical phenomena taking place in such a subnuclear region of the world clearly show quantum theoretic behavior, so that we are obliged to reconstruct the quantum theoretic notion of space-time.

Namely, in the present quantum field theory what is relevant for our epistemology may be how to deal with observables and their expectation values consistently, but not to argue the space-time structure itself as our physical objects. Nevertheless, we usually start from the classical notion of Minkowski space. This may be simply because of our timidity about abandoning the familialized traditional structure of the classical space-time even in such a region as a subnuclear world.

In this connection it may be noteworthy to review the monumental lecture by Riemann, 1854. Long before the advent of Einstein's theory of

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relativity Riemann described precisely the fundamental notion of space metric as follows:

> Nun scheinen aber die empirishcen Begriffe, in welchen die räumlichen Massbestimmungen gegründet sind, der Begriff des festen Körpers und des Lichtstrahls, im Unendlichkleinen ihre Gültigkeit zu verlieren; es ist also sehr wohl denkbar, dass die Massverhältnisse des Raumes im Unendlichkleinen den Voraussetzungen der Geometric nicht gemäss sind, und dies würde man in der Tat annehmen müssen, sobald sich dadurch die Erscheinungen auf einfachere Weise erklären liessen (Riemann, 1892).

If we interpret the above statement in the terminologies of modern physics, the metric of space has no a priori form, but should be attributed to the properties of interaction. For example, the classical notion of distance, Euclidean metric, may have some validity only under the assumption of the existence of rigid bodies.

In the beginning of his lecture Riemann mentioned explicitly the importance of a discrete manifold and suggested its topological study. Indeed the concept of continuity is a powerful tool to deal with approximation process in physics, but it can hardly be realized in modern physics.

In 1966 Yukawa proposed an atomistic quantum field theory, introducing small four-dimensional space-time domains D_i ($i = 1, \ldots, N$). There field operators φ are defined as set functions of D_i. There Yukawa imposed a certain restriction on the geometrical properties of the domains in order to prohibit a return to the conventional theory in the limit.

In the present paper we shall try to elaborate Yukawa's original idea by making use of a new kind of measure space structure for the space-time. Effects that appear in our formulation of S-matrix representation will be discussed.

2. A TOPOLOGY OF PHYSICAL SPACE

Since we have no a priori notion with respect to the structure of subnuclear space, we had better start from an abstract or general framework of space as far as possible. Let us denote our physical space-time and its topology by $\mathfrak{X}$ and $\mathfrak{T}$, respectively. $\mathfrak{X}$ may manifest an elementary-domain structure in the subnuclear region, while $\mathfrak{T}$ may reflect a certain quantum-theoretic property as creation and annihilation of particles in the region. Our aim is to construct the physical space $(\mathfrak{X}, \mathfrak{T})$ from the macroscopic space (X, τ), where τ is a standard topology. We shall call (X, τ) a reference space X and assume that it is normal and satisfies the second axiom of countability. This means that for any two disjoint closed sets there exist disjoint neighborhoods and any open set can be represented as a union of open sets which belong to a fixed countable family of open sets. In other words, (X, τ) has a continuity property, and is metrizable due to the Urysohn-Tychonoff theorem (Kelley, 1955), although we need not necessarily assume any metric explicitly.

The reference space X has been utilized to describe macroscopic phenomena, where the notion of continuity plays an important role for the conventional procedures of approximation. In fact, when we want to formulate approximation processes we have to assume the existence of limiting values in our framework.

Let us explain the reference space X and its topology in more detail. X is a certain set of elements which are not necessarily conventional points but basic entities for constructing macroscopic space. τ is a family of subsets of X satisfying the following conditions:

(T-1) $X, \phi \in \tau$, where ϕ is denotes an empty set.
(T-2) countable unions and finite intersections of subsets that belong to τ also belong to τ.

We shall call an element of τ an open set of X, and τ itself a standard topology. Obviously there are many ways to define such an open set even when the set X is definitely fixed. The reason why we start from such an abstract notion of topological space is to keep our freedom for choice of adequate physical models.

Furthermore, for the sake of conceptual convenience, i.e., to make our formulation complete and symmetric with respect to set-theoretical calculations $\cup$ and $\cap$, we extend the topology τ of X to a Borel field $\boldsymbol{\tau}$, which is a minimum σ algebra including τ.

Now we shall proceed to a decisive step towards a quantum theoretic topology of our space. As one of the widest frameworks of quantum theory we adopt the so-called C^* algebra $\mathfrak{A}$. Namely, dynamical variables or observables are assumed to be self-adjoint elements of a norm closed involutive ring with $*$-operation over complex field C including unity 1. Then what we have to do is to construct C^* algebra $\mathfrak{A}$ on the measurable space $(X, \boldsymbol{\tau})$.

Let A be an open set with finite volume with respect to a σ-finite measure m, i.e., $A \in \tau \subset \boldsymbol{\tau}$ and $m(A) < \infty$. C^* algebra $\mathfrak{A}(A)$ is generated by $\mu(A)$ and $\mu(A)^*$ which satisfy the following anticommutation relations:

$$\mu(A)\mu(A)^* + \mu(A)^*\mu(A) = m(A) \tag{2.1}$$

$$\mu(A)\mu(A) + \mu(A)\mu(A) = 0 \tag{2.2}$$

Thus we obtain a specified C^* algebra

$$\mathfrak{A} = \overline{\bigcup_{A} \mathfrak{A}(A)} \tag{2.3}$$

where the overbar means to take a closure.

At this stage we would like to introduce a new concept in order to relate a unique global observable with a topological and measurable structure of space-time which reveals itself through the behavior of field variables. We shall call it "supportance." A supportance of a Borel set B is defined as $\lambda(B) = \mu(B)^*\mu(B)$ which has σ additivity:

$$\lambda\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} \lambda(B_j) \tag{2.4}$$

for any disjoint family of Borel sets $\{B_j\}_{j=1}^{\infty}$.

The supportance of a Borel set B may be interpreted as a capacity of supporting field variables intrinsic to the subset B. This is a reason why we coin such a new terminology as supportance. Physically speaking, a concept of supportance is a kind of generalization or an abstraction of a particle number operator $n(D, t) = \bar{\varphi}(D, t)\varphi(D, t)$ introduced by Yukawa (1966). Indeed through the introduction of supportance, one may attribute a more general topology to the reference space X such that it can involve a discrete topology like elementary domains without referring explicitly to concrete geometrical structures of elementary domains. If we dare say, the supportance of a space-time region A may have a value proportional to a number of "elementary domains" contained in A.

Here, it should be noted that so far no symmetry condition like an invariance with respect to the Poincaré group has been imposed on our algebra $\mathfrak{A}$. This is simply because in the present paper we are concerned exclusively with the topological structure of space-time. We shall discuss later the symmetry properties of our framework.

To realize the physical significance of supportance more precisely, we introduce a physical *state* ω, which is defined by a normalized positive linear functional on $\mathfrak{A}$. Then an expectation value of a supportance of an open set A is given by $\omega(\lambda(A))$.

On the other hand, in virtue of anticommutation relations (2.1) and (2.2), we can derive an adequate spectral property of the supportance for any m-bounded set A as

$$\|\lambda(A)\| = m(A) \tag{2.5}$$

Since $\lambda(A)$ characterizes a quantum-theoretic topology of the physical space-time $\mathfrak{X}$ and can certainly involve a discrete topology, we shall call the new topology induced by $\lambda(A)$ a *quantized topology*. This is, in essence, similar to the complete set of elementary domains $\{n(D_1, t), \ldots, n(D_N, t)\}$ proposed by Yukawa (1966).

3. CONSTRUCTION OF A WHOLE PHYSICAL SPACE-TIME

In the previous section we focused our attention on the generalized topological structure of a physical space-time in the subnuclear region and introduced a quantized topology into the reference space-time. Then a question may arise as to the relation between the subnuclear region and the macroscopic ones. This has been always one of the most serious problems when we deal with the space-time structure to describe elementary particles and their fields in a comprehensive way. For example, if we try to apply straightforwardly a method of conventional general relativity to the subnuclear region, we encounter a serious drawback, namely, how to connect a curved inner space with a flat outer space consistently.

As is well known, the geometrical framework of general relativity has been physically concerned with the global structure of space-time. In fact, the local structure of the space has always been assumed to be flat in its limit, and various models of global structure have been constructed by differentiable metric tensors with certain connection coefficients. Here, we should remind ourselves of the fact that the space-time structure in the subnuclear region reveals itself only through quantum phenomena. So it seems hopeless for us to build a differential geometry in the level of classical physics. In this connection an attempt to construct quantum field theory on a stochastic basis has been made by Ingram (1962, 1964). But the difficulty with respect to comprehensive treatment of two spaces, inner and outer, is still an open question.

In the quantum field theory we deal only with observables of a certain dynamical system which manifest themselves in the space-time. Elementary particles created in an extremely localized region can travel a certain distance which is measurable in a macroscopic scale. Therefore kinematical behavior of elementary particles must be described in the classical framework of the Minkowski space, although quantum-theoretic treatment is needed for their observation.

In the circumstances it seems quite natural to assume that the reference space is on one hand endowed with the Minkowski space $\mathfrak{M}$ and the Lebesgue measure on it as a σ-finite measure. Then, carrying out the procedures similar to that explained in the last section, one can construct another C^* algebra $\mathfrak{A}_p = \cup_A \mathfrak{A}_P(A)$, where A is an arbitrary finite open set of $\mathfrak{M}$. $\mathfrak{A}_P(A)$ denotes the polynomial algebra of field operators whose supports are contained in A.

We are now ready to unify two fundamental notions, quantum field of matter and classical space-time, in a comprehensive scheme. As far as we are concerned with a physical reality elaborated in quantum mechanics, a physical space-time $\mathfrak{X}$ should be totally represented by a certain algebra of observables with respect to our physical objects, a quantum system. As we explained in the beginning of this paper, we adopted the C^* algebra for that purpose, because it seems at this moment to be one of the most promising models having the possibility of overcoming the difficulty related with the problem of an infinite degree of freedom.

Consequently we propose as a physical space the following tensor algebra:

$$\mathfrak{A}_T = \mathfrak{A}_P \otimes \mathfrak{A} \tag{3.1}$$

Here the tensor algebra means an algebra consisting of tensor products or Kronecker products of each element of $\mathfrak{A}_P(A)$ and each one of $\mathfrak{A}(A)$ such that it satisfies the distributive law.

It is straightforward to construct a S-matrix representation with respect to material fields in our scheme. For that purpose, of course, we ought to confirm the possibility of integration with respect to supportance λ. In virtue of several adequate properties imposed on the topology and measure of space $\mathfrak{X}$, we can uniquely define the integral. The proof is given in the Appendix. Let $\mathfrak{L}(x)$ be a Lagrangian density made of algebra $\mathfrak{A}_P$. We can write down the S matrix as

$$\begin{aligned} S &= P^* \exp\left[-\frac{i}{\hbar}\int \mathfrak{L}(x) \otimes \lambda(d^4x)\right] \\ &= P^* \exp\left[-\frac{i}{\hbar}\int \mathfrak{L}(x) \otimes \rho(x)\, d^4x\right] \end{aligned} \tag{3.2}$$

where $\rho(x)$ is a Radon–Nikodym derivative of λ and P^* means a procedure to take the covariant chronological order as in the usual quantum field theory.

An effect of the topological structure of space-time on the case of the quantized matter field $\varphi(x)$ may appear through commutation relations between $\mu(A)$ and $\mu(A)^*$. To specify the relations we need an entirely new physical notion, but the following may be possible:

$$\begin{aligned} [\mu^*(A), \varphi(x)]_\pm &= \delta_x(A) \\ [\mu(A), \varphi^*(x)]_\pm &= -\delta_x(A) \end{aligned}$$

where $\delta_x(\cdot)$ denotes the Dirac measure concentrated on x.

4. POSSIBLE CONSEQUENCES OF A NEW PHYSICAL SPACE-TIME

In the present paper we have argued mainly a general structure of the whole physical space-time. However, already in the S-matrix representation given by (3.2) some striking feature different from the conventional quantum field theory seems to have appeared. Let us explain the situation briefly.

Since the whole physical space-time $\mathfrak{X}$ is assumed to be the tensor algebra $\mathfrak{A}_T$, any physical state ω_T of the total quantum system should be represented by a tensor product of state functionals ω_P and ω defined on $\mathfrak{A}_P$ and $\mathfrak{A}$, respectively; namely,

$$\omega_T = \omega_P \otimes \omega \tag{4.1}$$

Naturally, what is relevant to the physical observation is an expectation value of the S matrix with respect to a state ω_T on $\mathfrak{A}_T$, i.e., the S-matrix element. It can be expressed as follows:

$$\begin{aligned}
\omega_T(S) &= (\omega_P \otimes \omega)\left\{P^* \exp\left[-\frac{i}{\hbar}\int \mathfrak{L}(x) \otimes \lambda(d^4x)\right]\right\} \\
&= 1 + \left(\frac{-i}{\hbar}\right) \int \omega_P(\mathfrak{L}(x))\omega(\lambda(d^4x)) \\
&\quad + \left(\frac{-i}{\hbar}\right)^2 \int \omega_P(P^*\{\mathfrak{L}(x)\mathfrak{L}(x')\})\omega(\lambda(d^4x)\lambda(d^4x')) \\
&\quad + \cdots
\end{aligned} \tag{4.2}$$

where the Dyson expansion (Bogoliubov and Shirkov, 1959) is used.

For an illustration of the feature mentioned above we may pick up the first integral appearing in the right-hand side of (4.2)

$$\int \omega_P(\mathfrak{L}(x))\omega(\lambda(d^4x)) \tag{4.3}$$

Since we have not specified concrete physical contents of two algebra $\mathfrak{A}$ and $\mathfrak{A}_P$, we are not in a position to prove the following conjecture, but it seems quite plausible: Namely, owing to the quantized topology λ, $\omega(\lambda(d^4x))$ is certainly not so smooth as the usual Lebesgue measure. It might be able to nullify singular regions of $\omega_P(\mathfrak{L}(x))$ by its zero contribution to the integral. Needless to say, the conventional quantum field theory corresponds to a special case of $\omega(\lambda(d^4x)) = d^4x$.

Previously several attempts have been made to eliminate the divergence difficulties by introducing stochastic notions to the fundamental metric tensors of four-dimensional pseudo-Riemannian space or quantizing the space-time. Concerning the former approach we may simply point out that the choice of randomness seems to be quite arbitrary and it can hardly be amalgamated with the present quantum theoretic scheme (Blokhintsev, 1973). On the other hand, the latter case is in appearance very similar to ours (Finkelstein, 1969).[2] Since the invariant volume element of the pseudo-

[2] In this paper the main stress is put on the quantum mechanical description of the causal relation between space-time points.

Riemannian space is given by $dV(x) = [-\det g(x)]^{1/2}\, d^4x$, the quantization of the gravitational field $g(x)$ may suggest to us a kind of supportance on the space as

$$\begin{aligned} \lambda(d^4x) &= \rho(x)\, d^4x \\ &= [-\det g(x)]^{1/2}\, d^4x \end{aligned} \tag{4.1}$$

However, in such an approach we may not be able to comprehend a whole physical space, because the structural connection of inner and outer world is not so clear.

There have been also two kinds of topological approaches to overcome the present difficulties of quantum field theory. One is to assign several intrinsic properties of elementary particles such as electric charges to certain characteristics which can be derived from a combinatorial topology of the Riemannian geometry (Wheeler, 1962). There the space is assumed to be a simply or multiply connected differentiable manifold. The other one is to access the atomistic properties of matter and energy by looking for a suitable discrete topology of the space-time world. The last approach has been suggested by Yukawa and developed by himself and his colleagues in the framework of "elementary domains" (Katayama and Yukawa, 1968; Katayama et al., 1968).

The present work, we think, is just on the same line as Yukawa's, but may be more general and abstract in its formalism, because we have tried to be free from any kind of model conceptualized in the macroscopic world as much as possible.

Of course, in pursuit of precise descriptions for various high energy phenomena we have to find out concrete forms of S-matrix representation and also to determine explicitly the state on the quantized topology of the space-time. Furthermore, a physical and mathematical interrelation between such a generalized topological and measurable structure of space-time and behaviors of field variables should be investigated more extensively. We shall discuss these problems in a forthcoming paper.

APPENDIX

For the sake of saving space we shall not repeat here the definitions of $\mathfrak{A}_P$, $\mathfrak{A}$, A, $\mathfrak{M}$, and λ, which are all given in the text.

Proposition. If $\mathfrak{L} \in \mathfrak{A}'_P \subset \mathfrak{A}_P$, then there exists the integral

$$\int_{\mathfrak{M}} \mathfrak{L}(x) \otimes \lambda(d^4x)$$

where subalgebra $\mathfrak{A}'_P$ consists of bounded field operators with compact supports.

Proof. Since $\mathfrak{A}_P$ is a norm closed algebra on a Lebesgue measure space, there exists a sequence of step functions $\{\mathfrak{L}_n\}_{n=1}^{\infty}$ which is Lebesgue integrable and uniformly converging to $\mathfrak{L}$ with respect to a norm $\|\cdot\|_P$ of $\mathfrak{A}_P$. The explicit form of $\mathfrak{L}_n$ is given as follows:

$$\mathfrak{L}_n(x) = \sum_{j=1}^{\infty} \mathfrak{L}_{n,j} \mathbb{1}_{A_{n,j}}(x)$$

where $\mathfrak{L}_{n,j} \in \mathfrak{A}_P(A_{n,j})$ and $\mathbb{1}_{A_{n,j}}$ denotes a characteristic function of $A_{n,j} \subset \mathfrak{M}$.

Since $\mathfrak{L}_n$ is Lebesgue integrable and (2.5) is satisfied, the following condition is always fulfilled:

$$\sum_{j=1}^{\infty} \|\mathfrak{L}_{n,j}\|_P \cdot \|\lambda(A_{n,j})\| < \infty$$

Hence one can define the integral of $\mathfrak{L}_n$ as

$$\int_{\mathfrak{M}} \mathfrak{L}_n(x) \otimes \lambda(d^4x) = \sum_{j=1}^{\infty} \mathfrak{L}_{n,j} \otimes \lambda(A_{n,j})$$

Let us examine the Cauchy condition of a sequence of integrals

$$\left\{\int_{\mathfrak{M}} \mathfrak{L}_n(x) \otimes \lambda(d^4x)\right\}_{n=1}^{\infty}$$

with respect to a norm $\|\cdot\|_T$ of the algebra $\mathfrak{A}_T = \mathfrak{A}_P \otimes \mathfrak{A}$. Using the distributive law satisfied by the tensor algebra $\mathfrak{A}_T$, Schwartz's inequality, and (2.5), successively, one can obtain the following relations:

$$\begin{aligned}
\left\|\int_{\mathfrak{M}} \mathfrak{L}_n(x) \otimes \lambda(d^4x) - \int_{\mathfrak{M}} \mathfrak{L}_m(x) \otimes \lambda(d^4x)\right\|_T &= \left\|\int_{\mathfrak{M}} \{\mathfrak{L}_n(x) - \mathfrak{L}_m(x)\} \otimes \lambda(d^4x)\right\|_T \\
&\leqslant \int_{\mathfrak{M}} \|\mathfrak{L}_n(x) - \mathfrak{L}_m(x)\|_P \cdot \|\lambda(d^4x)\| \\
&= \int_{\mathfrak{M}} \|\mathfrak{L}_n(x) - \mathfrak{L}_m(x)\|_P \, d^4x \\
&\to 0 \qquad \text{as } n, m \to \infty
\end{aligned}$$

Thus the existence of a limit of the sequence $\{\int_{\mathfrak{M}} \mathfrak{L}_n(x) \otimes \lambda(d^4x)\}_{n=1}^{\infty}$ is proved. The limit is denoted by

$$\int_{\mathfrak{M}} \mathfrak{L}(x) \otimes \lambda(d^4x)$$

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